III

RECTILINEAR DRAWING[1]

1. INTRODUCTION

SUPPOSE that one draws by means of a well-sharpened pencil many straight lines of uniform, narrow breadth extending across a sheet of white paper. What kinds of drawings may be made with this extremely restricted and artificial medium? Here we must hold the pencil point against the paper with a single definite pressure, while moving the point always at the same velocity.

More precisely, in the idealized form of the problem here considered, we take these straight lines to be of microscopic width, so as not to be individually discernible, and to be very numerous. The problem is then to determine whether or not a given (idealized) wash drawing can be reproduced by such (idealized) means, and further, just how it is arrived at.

Let us begin with a very simple illustration of such a problem: The given drawing is to be such that the surface density of lead deposited—determining the degree of blackness—is inversely proportional to the distance from a fixed point O. Can this particular drawing be made by such rectilinear means?

The answer is clearly affirmative. For imagine a very large number of lines drawn through O in an equiangular

[1]In a paper "On drawings composed of uniform straight lines" which has just appeared in Liouville's *Journal de mathématiques pures et appliqueés*, I have presented the same topic from a strictly mathematical point of view.

distribution (see fig. 1).[2] The amount of lead Q deposited within a circle of radius r having O as center is evidently proportional to r: $Q=kr$. Hence the amount of lead dQ in the ring between a circle of radius r and one of radius $r+dr$ (dr, an "infinitesimal") is $dQ=kdr$. But the area dA of this ring is $2\pi rdr$, since $A=\pi r^2$ is the area of a circle of radius r. Consequently the surface density of the lead is given by the ratio $dQ/dA=k/2\pi r$. Hence the density obtained is inversely proportional to the distance r from the fixed point O, as required.

There arises similarly in any such problem a fundamental density function F, depending upon position in the plane and corresponding to the degree of blackness of the drawing which ranges from white through gray to black. The extreme cases $F=0$ and $F=\infty$ correspond to white and black respectively. We may think of F as measured by the depth of the deposit of lead on the paper. Of course in actual practice not only has F a certain effective maximum, after which the lead does not adhere to the paper, but F will change gradually from point to point. However in our idealization of the problem we shall not always require F to be finite and continuous. Evidently in the special case considered above F becomes infinite at the point O.

On the other hand we shall always assume that the amount of lead laid down, $\int FdA$ (dA, element of area) is finite in any finite part of the plane.

An interesting variant of the general problem of rectilinear drawing specified above is obtained when we allow rectilinear erasures to be made after the drawing has been completed, with the natural requirement, of course, that no lines already drawn are to be erased. Here if F_d is the

[2]The drawings shown in this paper have been very kindly supplied by Mr. David Middleton, to whom I desire here to express my warm appreciation.

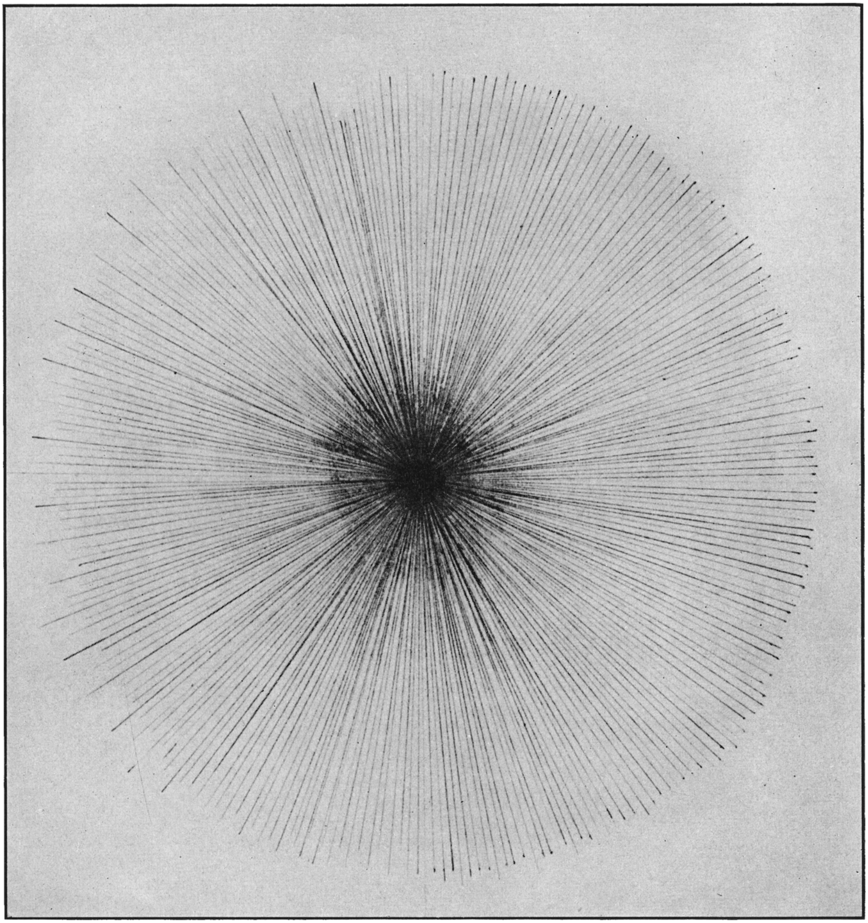


Fig. 1

density for the drawing, and F_e is the similar density corresponding to the erasure (so that $F_d \geqq F_e$ everywhere), then the final drawing clearly corresponds to the density function $F = F_d - F_e \geqq 0$.

A second variant of our problem is obtained if we allow only a single *uniform* erasure all over the plane, i.e., modify $F \geqq k > 0$ to $F - k$, where F is the density function for the given drawing and the constant k corresponds to the uniform erasure in question.

For our later purposes it is convenient to select a certain point O in the plane as center for a system of rectangular coordinates x, y and of related polar coordinates r, θ. Here r is regarded as positive or negative while θ is an angular coordinate of period 2π so that (r, θ) and $(-r, \theta+\pi)$ will

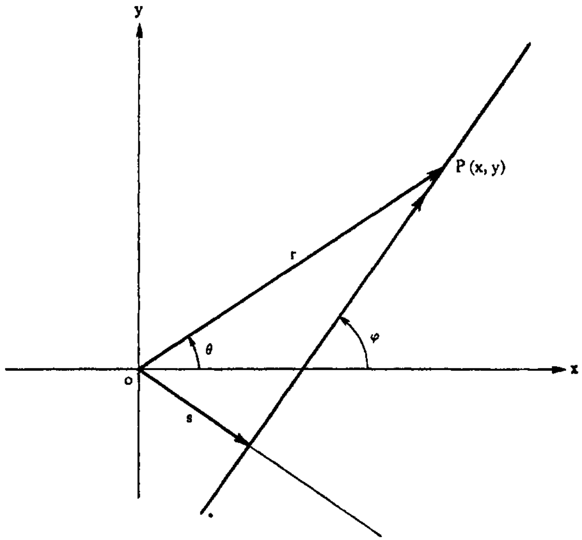


FIG. 2

denote the same point. Thus we may regard the density F as a function of r and θ of period 2π: $F = F(r, \theta)$, with $F(-r, \theta+\pi) = F(r, \theta)$. Similarly we may specify a line l

in the same plane by the coordinates s and φ as indicated in the figure. These are the usual line coordinates useful for our purposes.

2. ONE-PARAMETER FAMILIES OF STRAIGHT LINES

Let us imagine now that a family of straight lines be drawn which depends on a single parameter. We consider first the following case: the straight lines envelop a regular convex arc C with continuous curvature $1/\rho$, along which the points of tangency of the lines are distributed with an assigned frequency $f(s)$, where s designates arc length along the curve C (see figure 3).

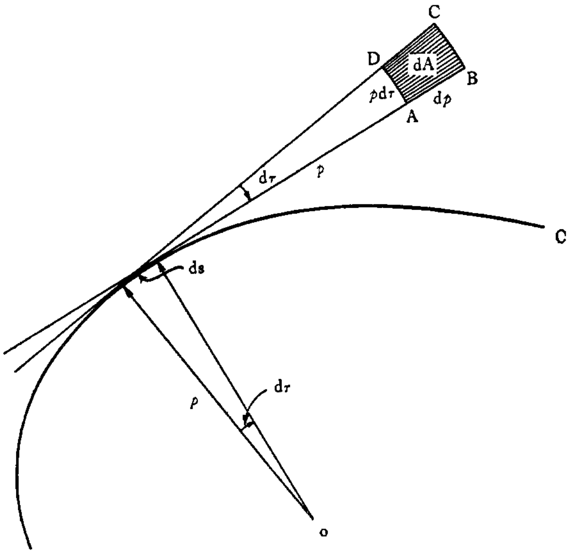


FIG. 3

This means that to get an approximation to the limiting form of drawing we may distribute the tangent lines along the curve C at points $f(s)\ ds$ apart, (ds, a small fixed increment of arc). It is obvious that the function $f(s)$ is a kind of distribution function, specifying the distribution of the given one-parameter family of straight lines.

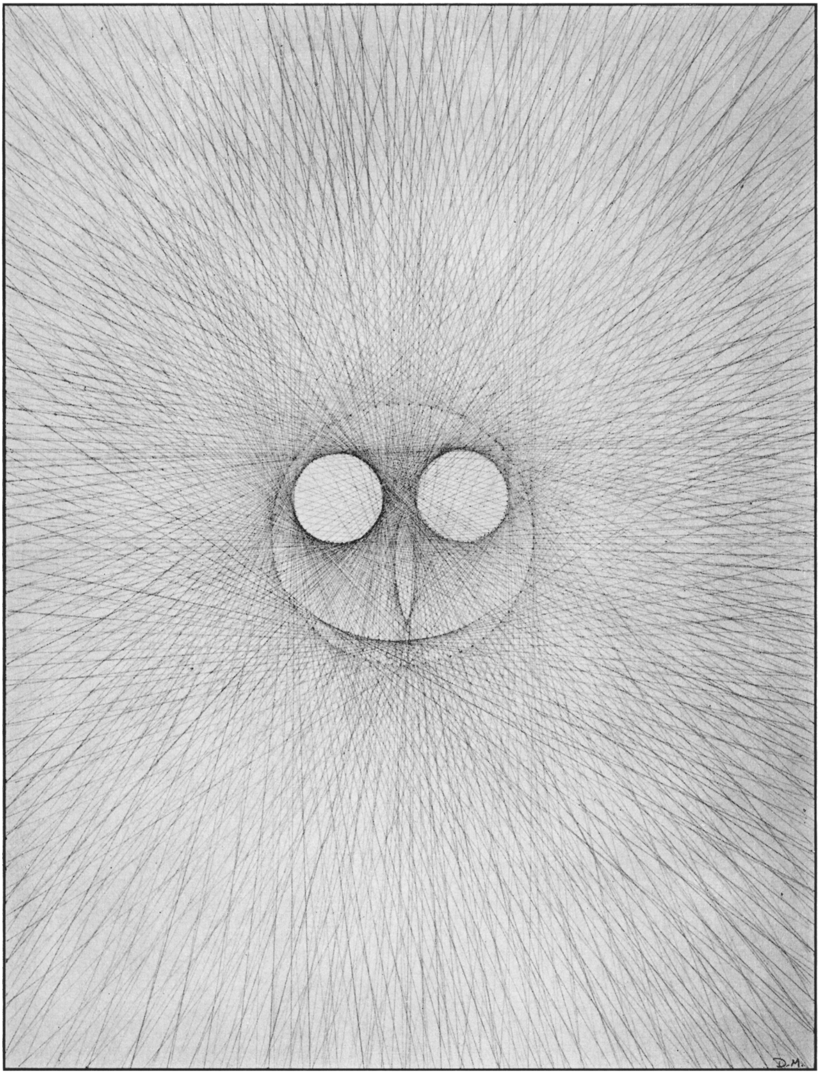


Fig. 4

On considering the above figure it is clear that the area dA of the small curvilinear quadrilateral $ABCD$ is given by $p d\tau dp$. Likewise the amount of lead in $ABCD$ is $f\rho d\tau dp$, since the length of each segment of the straight lines crossing $ABCD$ is dp while the number of these lines is $fds = f\rho d\tau$. Hence the limiting density at a point P is $f\rho/p$: $F = f\rho/p$.

Suppose now that we are given a drawing, and desire to determine whether or not it arises from such a one-parameter family of straight lines. According to what has just been shown, it is clearly necessary for this: (1) that the equation $F = \infty$ defines the convex arc; and (2) that along any tangent to this arc C, F varies inversely as the distance from the point of tangency. If these two conditions are satisfied and if $\rho(>0)$ denotes the radius of curvature at the point of tangency, the desired distribution function $f(s)$ is clearly Fp/ρ.

Before proceeding further it is worth while to note the practical use of such one-parameter families for obtaining outline drawings of curvilinear arcs C. In fact since the density F is infinite along the arc C, this curve appeared as clearly etched. Similarly any set of arcs can be simultaneously etched, although there is no reason to believe that the *shading* of masses will then be as desired. The pumpkin-head drawing of Mr. Middleton herewith shown (figure 4) affords a simple and amusing illustration of the possibilities.

It is interesting to observe that in such an outline drawing the density function F is given as a sum of terms of the form $f(s)\rho/p$ referred to above. For example, in his figure Mr. Middleton has taken $f(s)\rho$ to be the same along every circle and circular arc; this requirement means that the tangents are drawn at equal *angular* intervals along all of the circles and circular arcs, inasmuch as f is inversely proportional to

ρ. Hence in this case the density function at any point is simply given by the sum of the reciprocals of the lengths of tangents which can be drawn from the given point to the circles and circular arcs.

Our second case of a one-parameter family will be the degenerate case in which the curve C enveloped by the straight lines reduces to a single point. For definiteness we will further assume that the frequency of the points of intersection of the radial lines through this point O is given by a continuous function $f(\varphi)$ on the unit circle about the point O; here φ designates the angle which the lines make with a fixed line. It is then apparent that we have $F(r, \theta) = f(\varphi)/r$ as the corresponding density function, and that this density function is continuous everywhere except at the origin $r = 0$, where it is obviously discontinuous.

The third and last (one-parameter) case is the completely degenerate case in which there is no envelope and we have a family of parallel straight lines so that $F(r, \theta) = f(r \cos\theta)$ if all the lines make an angle of $\pi/2$ with the initial direction. If we had employed corresponding rectangular coordinates x, y, we would have

$$F(r, \theta) = F^*(x, y) = f(x),$$

or more generally, for an arbitrary direction of the family of parallel straight lines, we would have

$$F^*(x, y) = f(ax + by).$$

Here $f(ax + by)$ is taken as an arbitrary continuous function of its argument $ax + by$.

Now it has long been known that an arbitrary continuous function of x and y can be approximated to as a sum of continuous functions of such a linear variable $ax + by$. Consequently it is obvious that any positive continuous function can be approximated to by drawing suitable parallel families (or parts thereof) and then making suitable erasures of parallel families. More specifically, one need only draw the families of parallel lines appearing in the sum wherever the corresponding f is positive; and then make the necessary erasures wherever f is negative; in fact the two components have then the desired algebraic sum which is positive or zero.

Consequently we conclude that any drawing in the finite part of the plane corresponding to a continuous positive (or zero) density function can be approximated to by means of a finite set of continuous one-parameter families of parallel straight lines and by subsequent one-parameter families of parallel erasures.

3. TWO-PARAMETER FAMILIES OF STRAIGHT LINES

An arbitrary straight line l has been specified by two coordinates s and φ, of which the first is a radial coordinate, the second an angular coordinate of period 2π (see fig. 2); furthermore (s, φ) and $(-s, \varphi+\pi)$ then correspond to the same straight line. There is thus a two-parameter family of such straight lines.

When we attach a continuous two-parameter "distribution function" $f(s, \varphi)$ to such straight lines we mean that the mass due to the straight lines with an angle between φ and $\varphi+d\varphi$ and traversing a small area dA is nearly given by

$$f(s, \varphi)dAd\varphi,$$

where s denotes the distance of some such line through dA from the origin O.

More precisely, consider a series of n angles increasing from 0 to 2π through small equal increments $\Delta\varphi$ so that $n\Delta\varphi=2\pi$; and suppose that, for each such φ, for s increasing from 0 to S by small equal increments Δs so that $n\Delta s=S$, lines of breadth proportional to

$$f(\mu\Delta s,\, \nu\Delta\varphi)\Delta s\Delta\varphi \quad (\mu,\, \nu = 1, 2, \cdots n),$$

are drawn (i.e., $f(\mu\Delta s, \nu\Delta\varphi)$ lines of the same breadth). The kind of distribution of lines just specified is essentially independent of both the origin of coordinates and the direction of the initial line. It is clear that in a small area dA the limiting density as n increases without limit will be given by

$$\int_0^{2\pi} f(s, \varphi)d\varphi.$$

Now we have the relationship (see fig. 2) $s = r\sin(\varphi - \theta)$. In consequence we have the following general fundamental equation connecting the density function $F(r, \theta)$ and the distribution function $f(s, \varphi)$:

(1) $$F(r, \theta) = \int_0^{2\pi} f(r\sin(\varphi - \theta), \varphi)d\varphi.$$

Thus, given the continuous distribution function $f(s, \varphi)$, say for $|s| \leqq S$ $(S < +\infty)$ it is immediately possible to determine the corresponding continuous density function $F(r, \theta)$ for $|r| \leqq S$ by the above formula.

Our primary concern is of course the inverse problem: Given a density function $F(r, \theta)$ for $|r| \leqq R < +\infty$, to ascertain whether or not there exists a corresponding distribution function $f(s, \varphi)$ for $|s| \leqq R$, and further to determine all such functions (when they exist). From this point of view the fundamental equation (1) appears as a special type of "linear integral equation of the first kind" for the unknown function $f(s, \varphi)$. Since we are here concerned with determining whether or not a given wash drawing can be reproduced by means of smooth distributions in distance and angle of very many fine indefinitely extended straight lines, the solution $f(s, \varphi)$, as well as $F(r, \theta)$, is in general required

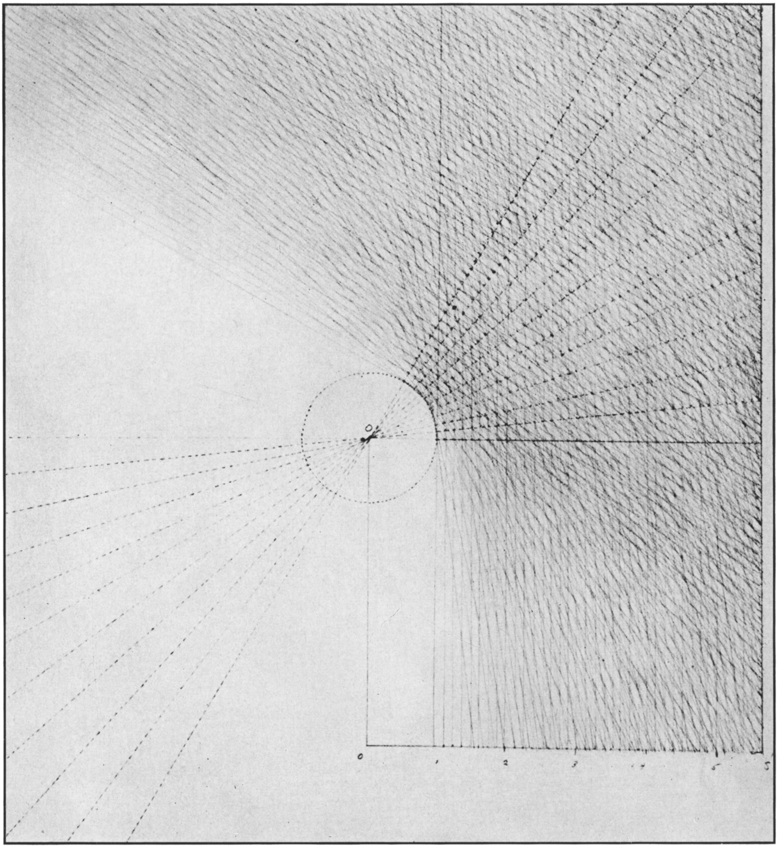


Fig. 5

to be continuous in its arguments. The singular distributions hitherto mentioned are of course not of this type.

One of the simplest possible types of two-parameter distributions, aside from the trivial case $f(s, \varphi) = 1$ when $F(r, \theta) = 2\pi$, is that in which $f(s) = 0$ for $s < 1$ and $f(s) = k$ for $s > 1$ (fig. 5).

Here it is readily found from (1) that the corresponding density function is

$$F(r, \theta) = \begin{cases} 0 \text{ for } r < 1 \\ 4\pi k\left(\frac{1}{2} - \frac{1}{\pi} \sin^{-1}\frac{1}{r}\right) \text{ for } r > 1. \end{cases}$$

When rectilinear erasures are not allowed (First Problem) f must be positive or zero everywhere; if we allow rectilinear erasures after the drawing has been made (Second Problem), the restriction that f is of one sign is abandoned; and if we allow only a single uniform erasure (Third Problem), we are interested in the positive solutions $f(s, \varphi)$ obtained when $F(r, \theta)$ is increased by a suitable positive constant.

4. THE SYMMETRIC CASE

A very interesting special case is that in which the density function $F(r, \theta)$ depends merely on the distance from a fixed point, say the origin O, so that $F(r, \theta) = F(r)$ with $F(-r) = F(r)$. This will be called the symmetric case for obvious reasons; the case just specified is of this type. It is natural to conjecture that in the symmetric case we may restrict attention to two-parameter distribution functions $f(s, \varphi)$ of like symmetric type: $f(s, \varphi) = f(s)$ with $f(-s) = f(s)$, i.e., corresponding to lines equally distributed in all directions about the origin.

The fact that we need only consider distribution functions of this symmetric type may be proved as follows.

Suppose first that a non-symmetrical continuous distribution function $f(s, \varphi)$, actually involving φ, does yield a symmetrical density function $F(r)$. It is then clear that $f(s, \varphi+c)$ for any c would yield the same density function, and thus that the average

$$[f(s, \varphi)+f(s, \varphi+\Delta\varphi)+\cdots+f(s, \varphi+(n-1)\Delta\varphi)]/n$$

with $n\Delta\varphi=2\pi$, would also. Proceeding to the limit, we see that the symmetrical distribution function

$$\bar{f}(s)=\int_0^{2\pi} f(s, \varphi)d\varphi/2\pi$$

would also yield the assigned density $F(r)$.

Hence if a non-symmetrical $f(s, \varphi)$ did exist, so would a symmetrical $\bar{f}(s)$.

As follows from a later result, there cannot exist such a non-symmetrical $f(s, \varphi)$. In fact if there did exist such an $f(s, \varphi)$, the difference $g(s, \varphi)=f(s, \varphi)-\bar{f}(s)$ would satisfy the homogeneous linear integral equation,

$$0=\int_0^{2\pi} g(r\sin(\varphi-\theta), \varphi)d\varphi.$$

If we write here $\varphi=\theta+\psi$, this equation takes the equivalent form

$$0=\int_0^{2\pi} g(r\sin\psi, \theta+\psi)d\psi.$$

But it will be proved that the basic equation (1) admits of a unique solution f at most; hence in the case $F=0$, the only possible one is trivial one $f=0$, and this is the equation under consideration with $f=g$. We observe also that if other solutions existed it would be possible to make a rectilinear drawing (non-uniform) and erase it all by rectilinear erasure along other lines. At present, however, we shall only prove that at most one symmetrical solution can exist. Otherwise we obtain the following still simpler equation:

$$\int_0^{2\pi} g(r \sin \psi) d\psi = 0$$

Thus we infer that the integral last written vanishes identically in r. Now introduce the variable $s = r \sin \psi$ in this integral. In this way we conclude that we must have

$$\int_0^r \frac{g(s)ds}{\sqrt{r^2 - s^2}} = 0.$$

This is a special case of a type of integral equation treated by Abel to which we shall refer later. Let us solve it explicitly by his simple, direct method, of which the generalization is immediate. Multiply this integral through by $r/\sqrt{\rho^2 - r^2}$ where $s < r < \rho$, and integrate as to r from 0 to ρ. We obtain thus

$$\int_0^\rho \left[\int_0^r \frac{rg(s)ds}{\sqrt{(\rho^2 - r^2)(r^2 - s^2)}} \right] dr = 0.$$

Making a valid interchange of the order of integration (see fig. 6) this becomes

$$\int_0^\rho g(s) \left(\int_s^\rho \frac{rdr}{\sqrt{(\rho^2 - r^2)(r^2 - s^2)}} \right) ds = 0.$$

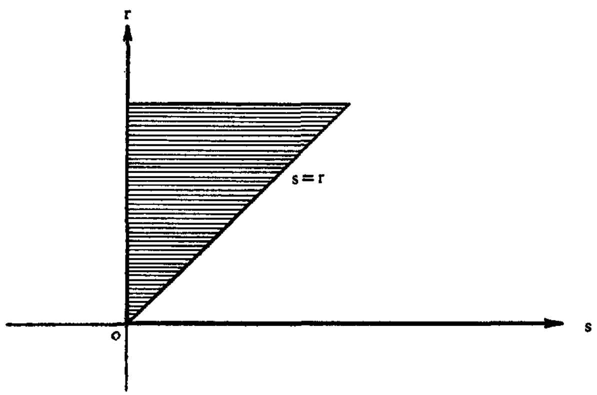


FIG. 6

But the inner definite integral has the value $\pi/2$. Hence we deduce

$$\int_0^\rho g(s)ds = 0;$$

and by differentiating as to ρ we conclude for all ρ and χ that $g(\rho)$ is zero. Thus g vanishes identically, contrary to assumption.

We see therefore that in the case of a continuous symmetrical density function $F(r)$ there is at most one corresponding continuous distribution function, f, which, if it exists, must likewise be symmetrical.

In consequence in the symmetric case we may take (1) in the more special form

$$F(r) = \int_0^{2\pi} f\big(r \sin(\varphi - \theta)\big)d\varphi = \int_0^{2\pi} f(r \sin \varphi)d\varphi,$$

where $F(-r) = F(r), f(-s) = f(s)$. Thus we need only to consider the equation

$$F(r) = 4\int_0^{\frac{\pi}{2}} f(r \sin \varphi)d\varphi,$$

or, making the change of variables $r \sin \varphi = s$,

$$F(r) = 4\int_0^r \frac{f(s)ds}{\sqrt{r^2 - s^2}}.$$

But this is precisely the integral equation solved by Abel (1828). In order to solve it we have only to multiply through by $r/\sqrt{\rho^2 - r^2}$ and integrate in r from 0 to ρ. This yields, as in the special case treated above,

$$2\pi\int_0^\rho f(s)ds = \int_0^\rho \frac{rF(r)dr}{\sqrt{\rho^2 - r^2}}.$$

If there is a continuous solution, the integral on the right must have a continuous derivative in ρ, namely $f(\rho)$.

Thus there exists such a continuous (symmetric) distribution function $f(s)$ if and only if the integral

$$I(s) = \int_0^s \frac{rF(r)dr}{\sqrt{s^2 - r^2}}$$

admits a continuous derivative, in which case we have necessarily

$$f(s) = \frac{1}{2\pi} \frac{d}{ds} \int_0^s \frac{rF(r)dr}{\sqrt{s^2 - r^2}} \tag{2}$$

as the unique solution.

If we do not wish to restrict attention to continuous functions F and f, various extensions are clearly possible. One of the simplest of these would be that in which F is taken integrable in the sense of Lebesgue over the given region, while f is similarly integrable. Here we should be led to require that the integral $I(s)$ not only exist but be absolutely continuous.

5. A SPECIAL SYMMETRIC CASE

We shall apply the preceding formal work to the discussion of a particular symmetric case which is especially interesting, namely,

$$F(r) = \begin{cases} 0 \text{ for } r < r_0 \\ k \text{ for } r > r_0 \end{cases}.$$

Here the function $F(r)$ is discontinuous at $r = r_0$ but not in a way such as to cause essential difficulty. In fact if we apply the formula (2) to this case we find at once

$$f(s) = \begin{cases} 0 \text{ for } s < r_0 \\ \dfrac{k}{2\pi} \quad \dfrac{s}{\sqrt{s^2 - r_0^2}} \text{ for } s > r_0. \end{cases}$$

Since the function $f(s)$ so obtained is everywhere positive and leads to no difficulty in (1), we conclude that it is possible to set up a symmetrical distribution of lines outside of the circle $s = r_0$ in such wise that the region outside of the given circle is of a *uniform gray.*

In the accompanying drawing (fig. 7) Mr. Middleton takes only sixteen lines for each radial direction and lets the angle increase successively by 5°. His result is shown herewith and evidently accomplishes the desired result as far as could be expected in view of the relatively few lines used for each direction.

If, on the other hand, we take a good many lines for each direction, these directions being at a considerable angle (say $\pi/4$ radians) apart, another interesting type of approximation is obtained (fig. 8).

Since a circular white spot is the "negative" of a dot, it is clear that we can draw a "faint negative" of any drawing by a process of stippling. More precisely, we can reproduce a doubly exposed negative obtained from a first weak exposure to a stippled reproduction of the given drawing and a subsequent strong exposure to uniform light.

6. THE GENERAL HARMONIC CASE

Imagine now a continuous distribution function $f(s, \varphi)$ which is a simple harmonic function of φ of period $\frac{2\pi}{m}$:

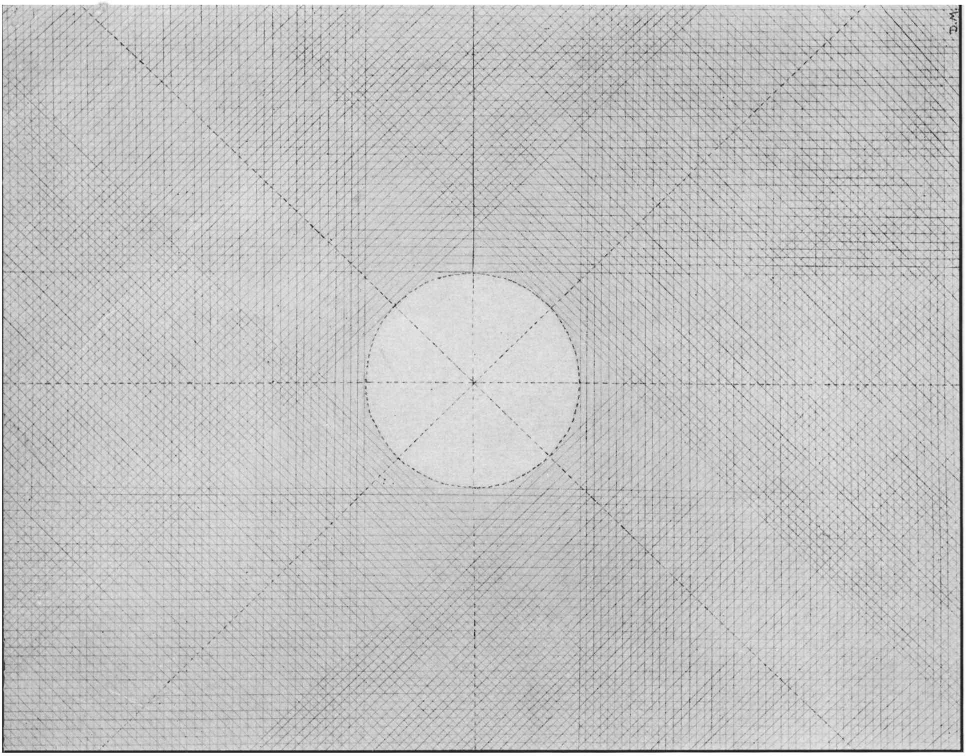
$$f_m(s, \varphi) = f_m(s) \cos m\varphi + g_m(s) \sin m\varphi.$$

Here $f_m(s)$, $g_m(s)$ are to be regarded as even or odd according as m is even or odd, so that the functional identity $f_m(-s, \varphi+\pi) = f_m(s, \varphi)$ holds and $f_m(0) = g_m(0) = 0$.

We shall speak of such an $f_m(s, \varphi)$ as harmonic of the mth order ($m \geqq 0$); in particular, the symmetric case is the harmonic case of zero-th order.

A density function $F(r, \theta)$ will similarly be said to be harmonic of the mth order if

$$F(r, \theta) = F_m(r) \cos m\theta + G_m(r) \sin m\theta.$$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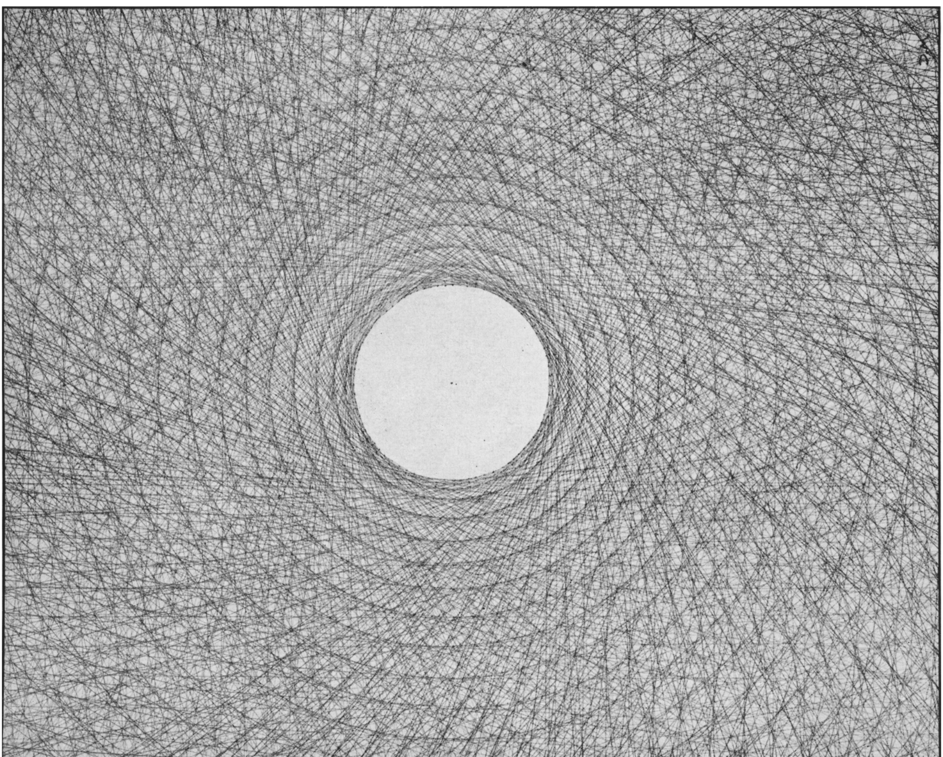

It is easily established that such a continuous harmonic distribution function of the mth order $f(s, \varphi)$ yields always a corresponding continuous harmonic distribution function of the mth order, $F(r, \theta)$.

In fact, from (1) we obtain directly

$$
\begin{aligned}
F(r, \theta) &= \int_0^{2\pi} (f_m(r \sin(\varphi - \theta)) \cos m\varphi \\
&\quad + g_m(r \sin(\varphi - \theta)) \sin m\varphi) d\varphi \\
&= \int_0^{2\pi} (f_m(r \sin \chi) \cos m(\chi + \theta) \\
&\quad + g_m(r \sin \chi) \sin m(\chi + \theta)) d\chi \\
&= \left[\int_0^{2\pi} (f_m(r \sin \chi) \cos m\chi \right. \\
&\quad \left. + g_m(r \sin \chi) \sin m\chi)\, d\chi \right] \cos m\theta \\
&\quad + \left[\int_0^{2\pi} (-f_m(r \sin \chi) \sin m\chi \right. \\
&\quad \left. + g_m(r \sin \chi) \cos m\chi)\, d\chi \right] \sin m\theta.
\end{aligned}
$$

Conversely we can at least conclude that if for a given harmonic density function $F_m(r, \theta)$ of order m there is a corresponding continuous distribution function $f(s, \varphi)$, this will necessarily be unique and also harmonic of order m.

To establish this uniqueness, suppose that $f(s, \varphi)$ is a distribution function yielding the harmonic $F_m(r, \theta)$ as corresponding density function. The expansion of $f(s, \varphi)$ in a Fourier series breaks $f(s, \varphi)$ up (formally) into an infinite number of harmonic components of orders $0, 1, 2, \cdots$, which, by what has just been proved, are carried over into the Fourier components of $F(s, \varphi)$ of the same orders. Consequently if we can prove that a harmonic distribution function can only be carried into the density function 0 if the distribution function itself vanishes identically, we will have proved that $f(s, \varphi)$ is harmonic of the order m in question. Furthermore it will follow that the function $f(s, \varphi)$, if it exists, is unique.

Thus we need only prove that if a harmonic distribution function of order m, $f_m(s, \varphi)$, is not identically zero it cannot yield a harmonic density function $F_m(r, \theta)$ which is identically zero. This has already been proved for $m=0$. But for $m>0$ we see that $F_m(r, \theta)=0$ would imply

$$\int_0^{2\pi} f_m(r \sin \chi) \cos m\chi d\chi = \int_0^{2\pi} g_m(r \sin \chi) \sin m\chi d\chi = 0$$

for $f_m(s)$ and $g_m(s)$ not both identically zero.

We will only prove this to be impossible for the cases $m=1,2$ since the extension can then be made at once to the cases $m=3,4 \cdots$. Suppose that we have for $m=1$

$$\int_0^{2\pi} f_1(r \sin \chi) \cos \chi d\chi = 4\int_0^{\frac{\pi}{2}} f_1(r \sin \chi) \cos \chi d\chi = 0.$$

Multiply through by r and integrate; we obtain at once $f_1^{(-1)}(r)=0$, where we write

$$f_1^{(-1)}(u) = \int_0^u f_1(u) du.$$

Hence we infer $f_1(r) \equiv 0$. Likewise if we have

$$\int_0^{\frac{\pi}{2}} g_1(r \sin \chi) \sin \chi d\chi = 0,$$

then by multiplying through by dr and integrating from 0 to r,

$$\int_0^{\frac{\pi}{2}} g_1^{(-1)}(r \sin \chi) d\chi = 0.$$

This is a homogeneous equation of the Abel type in $g_1^{(-1)}(s)$, and we find similarly

$$g_1^{(-1)}(s) \equiv 0, \text{ and so } g_1(s) \equiv 0.$$

We pass now to the case $m=2$. Suppose that we have

$$\int_0^{2\pi} f_2(r \sin \chi) \cos 2\chi d\chi = 0,$$

which we write in the form

$$\int_0^{2\pi} f_2(r \sin \chi)(\cos \chi \cos \chi - \sin \chi \sin \chi)d\chi = 0.$$

On integrating by parts the first term evidently takes the form

$$\frac{1}{r}\int_0^{2\pi} f_2^{(-1)} (r \sin \chi) \sin \chi d\chi,$$

while the second is seen to be equal to

$$-\frac{d}{dr}\int_0^{2\pi} f_2^{(-1)} (r \sin \chi) \sin \chi d\chi.$$

Hence if we write

$$W = \int_0^{2\pi} f_2^{(-1)} (r \sin \chi) \sin \chi d\chi,$$

we conclude that

$$\frac{1}{r}W - W' \equiv 0,$$

i.e., $W = cr$. But since $W'(0) = 0$ we must have $c = 0$, and so $W \equiv 0$.

Employing now the same argument as in the case $m = 1$, for the equation $W = 0$ we find that $f_2^{(-1)}(s) = 0$ and so $f_2(s) \equiv 0$.

Likewise we readily infer that $g_2(s) \equiv 0$.

More generally, we may obtain the stated result for $m = 1, 2 \cdots$ in succession, by writing at the mth stage

$$\cos m\chi = \cos \chi \cos (m-1)\chi - \sin \chi \sin (m-1)\chi$$
$$\sin m\chi = \sin \chi \cos (m-1)\chi + \cos \chi \sin (m-1)\chi$$

in the equations involving $f_m(s)$ and $g_m(s)$ respectively, and thus reducing the question to one of the type considered at the $(m-1)$st stage.

In this manner the stated result is readily established.

We are now in a position to formulate preliminary necessary and sufficient conditions for a continuous solution of the general harmonic case of the mth order.

A necessary and sufficient condition that, to a general harmonic continuous density function of the mth order,

$$F(r, \theta) = F_m(r) \cos m\theta + G_m(r) \sin m\theta,$$

there corresponds a continuous distribution function [unique and harmonic of the same order m by what precedes],

$$f(s, \phi) = f_m(s) \cos m\phi + g_m(s) \sin m\phi$$

is that the integral equation for $h_m(s)$,

$$H_m(r) = \int_0^{2\pi} h_m(r \sin u) e^{-imu} du \tag{3}$$

where

$$H_m(r) = F_m(r) + iG_m(r),$$

admits of a continuous solution, whose real and imaginary coefficients will then yield $f_m(s)$ and $g_m(s)$ respectively.

The proof of this italicized statement follows immediately from (1) and what has been proved in the preceding section.

7. EXPLICIT SOLUTION IN THE HARMONIC CASE $m = 1$

In the harmonic case $m = 1$ we have to consider the integral equation

$$H_1(r) = \int_0^{2\pi} h_1(r \sin u) e^{-iu} du, \tag{3_1}$$

which expresses the relation between the known coefficients $F_1(r)$, $G_1(r)$ of the density function F and the like coefficients $f_1(s)$, $g_1(s)$ of the distribution function $f(s, \phi)$ which it is desired to find. This equation may be written

$$H_1(r) = \int_0^{2\pi} h(r \sin u)(\cos u - i \sin u) du.$$

But the first component of the integral on the right is clearly

$$\frac{1}{r} h_1^{(-1)} (r \sin u) \Big|_0^{2\pi} = 0,$$

while the second term evidently has the value

$$-i\frac{d}{dr}\int_0^{2\pi} h_1^{(1)}(r\sin u)du = -4i\frac{d}{dr}\int_0^{\frac{\pi}{2}} h^{(-1)}(r\sin u)du.$$

Thus the equation under consideration is equivalent to

$$\frac{i}{4}\int_0^r H_1(r)dr = \int_0^{\frac{\pi}{2}} h^{(-1)}(r\sin u)du,$$

at least if the function $H_1(r)$ is suitably restricted near $r=0$. For example, it would suffice if we assumed that $H_1(r)$ remains finite near $r=0$. But the right-hand member is evidently the same as

$$\int^r h_1^{(-1)}(s)\frac{ds}{\sqrt{r^2-s^2}},$$

so that the equation is essentially a linear integral equation of Abel type for $h_1^{(-1)}(s)$. Hence we obtain

$$h_1^{(-1)}(s) = \frac{i}{4\pi}\frac{d}{ds}\left[\int_0^s \frac{r(\int_0^r H_1(r_1)dr_1)dr}{\sqrt{s^2-r^2}}\right],$$

where the outer integral in the right-hand member clearly vanishes for $s=0$ and has a continuous first derivative. Hence this equation is equivalent to

$$h_1(s) = \frac{i}{2\pi}\ \frac{d^2}{ds^2}\left[\int_0^s \frac{r(\int_0^r H_1(r_1)dr_1)dr}{\sqrt{s^2-r^2}}\right].$$

Thus we infer that a necessary and sufficient condition for a solution to exist is that the integral on the right side of the equation just written admits a continuous second derivative, in which case the unique solution is provided by the same equation.

But the double integral on the right may clearly be written in inverse order of integration as

$$\int_0^s H_m(t)\left(\int_t^s \frac{rdr}{\sqrt{s^2-r^2}}\right)dt = \int_0^s H_m(t)(s^2-t^2)^{\frac{1}{2}}dt,$$

so that

$$h_1(s) = \frac{i}{2\pi}\,\frac{d^2}{ds^2}\left[\int_0^s H_1(t)(s^2-t^2)^{\frac{1}{2}}dt\right].$$

Further, we may differentiate once *under* the integral sign so that $(s^2-t^2)^{\frac{1}{2}}$ is replaced by the derivative of this expression. If we write $s = tu$ we see, however, that if $u = s/t > 1$,

$$\frac{d}{ds}(s^2-t^2)^{\frac{1}{2}} = \frac{d}{du}(u^2-1)^{\frac{1}{2}}.$$

Hence we obtain the final solution for $m = 1$:

$$(4)\qquad h_1(s) = \frac{i}{4\pi}\,\frac{d}{ds}\left[\int_0^s H_1(t)\frac{2u}{\sqrt{u^2-1}}dt\right],\quad \left(u = \frac{s}{t} > 1\right).$$

Thus our general conclusion in the case of a harmonic distribution of the first order is as follows: *In the case of a given harmonic density function of the first order* $(m = 1)$

$$F_1(r,\,\theta) = F_1(r)\cos\theta + G_1(r)\sin\theta,$$

where $H_1(r) = F_1(r) + i\,G_1(r)$ *remains finite near* $r = 0$,[3] *there exists a corresponding continuous distribution function,*

$$f_1(s,\,\varphi) = f_1(s)\cos\varphi + i\,g_1(s)\sin\varphi,$$

likewise bounded near $s = 0$ *and harmonic of the first order, if and only if the integral in* (4) *represents a function with a continuous derivative. In this event the unique solution* $h_1(s)$ *is exhibited in* (4), *where of course,*

$$h_1(s) = f_1(s) + i\,g_1(s).$$

9. THE SOLUTION IN HIGHER HARMONIC CASES $m > 1$

The method of solution used above applies essentially to any order $m > 1$. Here we start by writing the identity

[3]We are assuming that $F(r,\theta)$ is continuous for $r>0$ only.

$$e^{-im\varphi} = (\cos \varphi - i \sin \varphi) e^{-i(m-1)\varphi},$$

and then proceeding essentially as in the case $m=1$ treated above.

We therefore content ourselves with stating the final result and giving in an appended Note the series of reductions involved. The reader who wishes a more detailed deduction will find it in my article referred to at the outset.

In the case of a given harmonic distribution function of the mth order,

$$F_m(r, \theta) = F_m(r) \cos m\theta + G_m(r) \sin m\theta,$$

where $H_m(r) = F_m(r) + i G_m(r)$ *is supposed to vanish to the* $(m-1)$*st order at* $r=0$ *so that* $H_m(r)/r^{m-1}$ *remains finite, there exists a corresponding continuous distribution function.*

$$f_m(s, \varphi) = f_m(s) \cos m\varphi + g_m(s) \sin m\varphi,$$

likewise harmonic of the mth order, if and only if the integral in the equation

$$(5) \quad h_m(s) = \frac{i^m}{4\pi} \frac{d}{ds} \int_0^s H_m(t) \frac{(u+\sqrt{u^2-1})^m + (u-\sqrt{u^2-1})^m}{\sqrt{u^2-1}} dt,$$

$$\left(u = \frac{s}{t} > 1\right),$$

is continuous and admits a continuous derivative as to s, *in which case the unique solution is derived by setting* $h_m(s)$ *in* (5) *equal to* $f_m(s) + i g_m(s)$.

10. EXPLICIT SOLUTION IN THE NON-HARMONIC CASE

In virtue of the known relationship between continuous functions and their Fourier series we are now in a position to formulate the general solution of our problem:

If the given continuous density function $F(r, \theta)$ *be expanded in a Fourier series*

$$\tfrac{1}{2}F_o(r)+\sum_{m=1}^{\infty}(F_m(r)\cos m\theta+G_m(r)\sin m\theta),$$

and if $F_m(r)$ *and* $G_m(r)$ *vanish to the*$(m\text{-}1)$*th order in* r *for all* m,[4] *then there will exist a corresponding continuous distribution function if and only if the integrals*

$$I_m(s)=\int_0^s(F_m(t)+iG_m(t))\frac{(u+\sqrt{u^2-1})^m+(u-\sqrt{u^2-1})^m}{\sqrt{u^2-1}}dt \qquad \left(u=\frac{s}{t}>1\right), \tag{6}$$

are continuous for $s\geqq 0$ *together with their first derivatives in such wise that the continuous functions* $f_m(s)$, $g_m(s)$ *defined by*

$$f_m(s)+ig_m(s)=\frac{i^m}{4\pi}I'_m(s) \tag{7}$$

form the Fourier coefficients of a continuous function $f(s,\varphi)$ *corresponding to*

$$\tfrac{1}{2}f_o(s)+\sum_{m=1}^{\infty}(f_m(s)\cos m\varphi+g_m(s)\sin m\varphi).$$

In this case $f(s,\varphi)$ *forms the unique continuous solution of the distribution problem.*

The proof is readily made by means of an induction based on the equations of the Note appended to the present paper.[5]

II. ON THE THREE DRAWING PROBLEMS

On the basis of what precedes we obtain immediately the following general conclusions for the three drawing problems specified earlier:

(1) A drawing with given continuous[6] density function $F(r,\theta)\geqq 0$ can be made without rectilinear erasures by means

[4]This condition will be satisfied, for instance, if $F(r,\theta)$ has continuous partial derivatives in x and y of all orders k in open continua S_k containing the origin.

[5]See also my paper already referred to.

[6]That is, continuous in x and y.

of a suitable continuous[7] two-parameter distribution function $f(s, \phi)$ if and only if $f(s, \varphi)$ exists as specified above and is positive or zero. If such a solution exists it will be unique.

(2) A drawing with given density function $F(r, \theta) \geqq 0$ can be made by means of a continuous two-parameter continuous distribution function $f_1(s, \varphi) \geqq 0$, followed by a similar distribution function $f_2(s, \varphi)$ of non-overlapping rectilinear erasures, if and only if $f(s, \varphi)$ exists as specified above, when we may take

$$f_1(s, \varphi) = \Big(|f(s, \varphi)| + f(s, \varphi)\Big)/2, \; f_2(s, \varphi) = \Big(|f(s, \varphi)| - f(s, \varphi)\Big)/2$$

so that

$$f(s, \varphi) = f_1(s, \varphi) - f_2(s, \varphi).$$

If such a solution exists it is unique, and evidently requires the least possible drawing and subsequent erasure.[8]

(3) A drawing with given continuous density function $F(r, \theta) \geqq 0$ can be made by means of a continuous two-parameter distribution function $f(s, \varphi)$ followed by a single *uniform* erasure if and only if $f(s, \varphi)$ exists as specified above. If $f(s, \varphi)$ is positive or zero everywhere no subsequent erasure is of course necessary. If, however, $f(s, \varphi)$ has a negative minimum $-m$, we first make the drawing with positive density function $F(r, \theta) + 2\pi m$ and corresponding distribution function $f(s, \varphi) + m \geqq 0$, and then make a uniform erasure with density $2\pi m$. Evidently this solution is essentially unique, and requires the least possible uniform erasure.

It is interesting to note that in the first problem when no erasures are allowed, certain obvious geometrical conditions must be satisfied if the drawing is to be possible. In order to

[7]That is, continuous in s and φ.
[8]I.e., as measured by lead put down and lead erased.

illustrate this fact as simply as possible let us restrict attention to the symmetric case, $F=F(r)$, when, as we have seen, the distribution function must also be symmetric, $f=f(s)$.

If we consider two circles C_a and C_b of radius a, b with $a>b>0$, concentric with the origin it is clear that the ratio of the length of any chord of C_a to the length of the part of the chord within C_b is at least a to b. It follows that the amount of lead $2\pi\iint F(r,\theta)rdrd\theta$ within C_a is at least a/b times that within C_b, so that we must have

$$\frac{1}{a}\int_0^a F(r)rdr \geqq \frac{1}{b}\int_0^b F(r)rdr \qquad (a>b),$$

if the drawing is to be possible without rectilinear erasures. This conclusion may of course also be derived directly from the fundamental equation (1) above, under the assumption $f(s,\varphi)\geqq 0$, the proof involving nothing more than an application of the geometric fact about chords stated above.

APPENDED NOTE

To treat the general case we start from the equations

$$(1) \qquad H_m(r)=\int_0^{2\pi} h_m(r\sin u)e^{-imu}du \qquad (m=0,1,2,\cdots)$$

where the $H_m(r)$ are assumed to be continuous for $r>0$ and vanish with r to the $(m-1)$st order. These equations may be written

$$(2) \qquad H_m(r)=\int_0^{2\pi} h_m(r\sin u)\ (\cos u - i\sin u)e^{-i(m-1)u}du$$
$$=\frac{i(m-1)}{r}\int_0^{2\pi} h_m^{(-1)}\,(r\sin u)e^{-i(m-1)u}du$$
$$-i\frac{d}{dr}\left[\int_0^{2\pi} h_m^{(-1)}(r\sin u)e^{-i(m-1)u}du\right],$$

where we define

$$(3) \qquad h_m^{(-1)}(s)=\int_0^s h_m(s)ds, \quad h_m^{(-2)}(s)=\int_0^s h_m^{(-1)}(s)ds,\cdots$$

If we multiply through by ir^{-m+1} and integrate from 0 to r, there results

$$(1') \qquad ir^{m-1}\int_0^r \frac{H_m(t)}{t^{m-1}}dt = \int_0^{2\pi} h_m^{(-1)}(r \sin u)e^{i(m-1)u}du,$$

since the right-hand member vanishes to order m at least in r. But (1′) is of a form like (1) on the right except that m is replaced by $m-1$. Repeating the same type of procedure $m-1$ times, we obtain finally

$$(4) \qquad i^m\int_0^r\left[\int_0^{t_m}t_{m-1}\cdots\left(\int_0^{t_1}\frac{H_m(t)}{t^{m-1}}dt\right)\cdots\right]dt_m = \int_0^{2\pi} h_m^{(-m)}\,(r \sin u)du.$$

If now we replace the m-fold integral on the left by a simple integral, (4) becomes

$$(5) \qquad \frac{2\left(\frac{i}{2}\right)^m}{(m-1)!}\int_0^r H_m(t)(r^2-t^2)^{m-1}dt = \int_0^{2\pi} h_m^{(-m)}(r \sin u)du.$$

But $h_m^{(-m)}(r)$ is even in r so that the integral on the right in (4) is four times the same integral taken between 0 and $\pi/2$. Writing $s = r \sin u$ we thus obtain, as the equivalent of (4), a linear integral equation of Abel type in $h_m^{(-m)}(s)$ with explicit solution

$$h_m(s) = \frac{1}{\pi}\,\frac{\left(\frac{i}{2}\right)^m}{(m-1)!}\,\frac{d^{m+1}}{ds^{m+1}}\left[\int_0^r r\left(\int_0^r\frac{H_m(t)}{t^{m-1}}(r^2-t^2)^{m-1}dt\right)\frac{dr}{\sqrt{s^2-r^2}}\right].$$

Conversely, if $h_m(s)$ as thus defined exists and is continuous, the equation (1) will be satisfied by $h_m(s)$.

But the right-hand integral, after inverting the order of integration, becomes

$$\int_0^s \frac{H_m(t)}{t^{m-1}}\left[\int_t^s \frac{(r^2-t^2)^{m-1}}{\sqrt{s^2-r^2}}\,r\,dr\right]dt,$$

so that we have

$$(6)\quad h_m(s)=\frac{1}{2\pi}\,\frac{i^m}{1\cdot3\cdot\cdots\cdot2m-1}\,\frac{d^{m+1}}{dr^{m+1}}\int_0^s \frac{H_m(t)}{t^{m+1}}(s^2-t^2)^{m-\frac{1}{2}}\,dt.\Big]$$

Now we may differentiate m times as to s *under* the integral sign because the integral and its first $m-1$ derivatives vanish for $t=s$. Writing then $s=tu$ in that $u\geqq1$ we obtain the equivalent form

$$(6')\quad h_m(s)=\frac{1}{2\pi}\,\frac{i^m}{1\cdot3\cdots2m-1}\,\frac{d}{ds}\int_0^s H_m(t)\frac{d^m}{du^m}(u^2-1)^{m-\frac{1}{2}}dt.$$

But it is readily proved by induction that

$$(7)\quad \frac{d^{m-1}}{du^{m-1}}(u^2-1)^{m-\frac{1}{2}}=\frac{1\cdot3\cdot\cdots\cdot2m-1}{2m}\left[(u+\sqrt{u^2-1})^m-(u-\sqrt{u^2-1})^m\right].$$

Substituting, we obtain the stated final explicit formula

$$(8)\quad h_m(s)=\frac{i^m}{4\pi}\frac{d}{ds}\left[\int_0^s H_m(t)\frac{(u+\sqrt{u^2-1})^m+(u-\sqrt{u^2-1})^m}{\sqrt{u^2-1}}dt\right]$$

$$\left(u=\frac{s}{t}>1\right)$$

yielding the explicit Fourier coefficient $f_m(s)$, $g_m(s)$ for $f(s,\varphi)$ in virtue of the formula $h_m(s)=f_m(s)+ig_m(s)$.

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